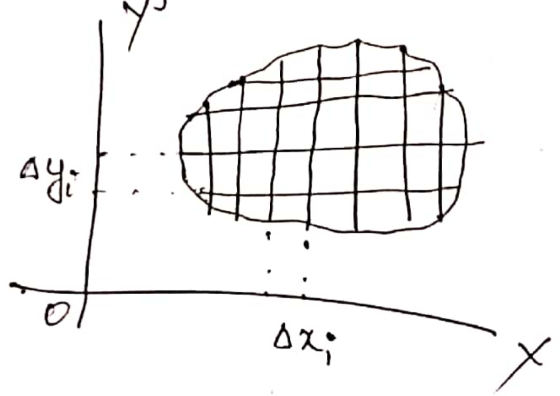
Double Integral

Definition:

Let $f(x,y)$ be a continuous single valued f^n of x and y within the region R bounded by the curve C. Let R be divided into n subinterval / regions i.e., $\Delta A_1, \Delta A_2, \ldots \ldots \Delta A_n$. Let (x_k, y_k) be any point in the region ΔA_k and consider the sum $\sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$. The limit of this sum as $n \to \infty$ and $\Delta A_k \to 0$ is defined as double integral of $f(x,y)$ over the region R and is defined or written as $\iint_R f(x,y)\, dA$



Evalvation of double

Double integral can be evaluated by expressing it in terms of two single integrals called iterated or repeated integral.

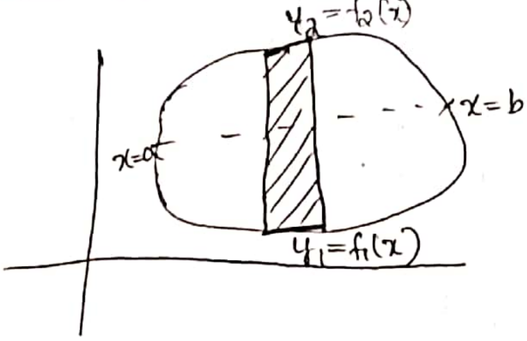
If R is described as $f_1(x) \le y \le f_2(x)$ i.e., $y_1 \le y \le y_2$ and $a \le x \le b$ then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\, dx\, dy$$

Method I

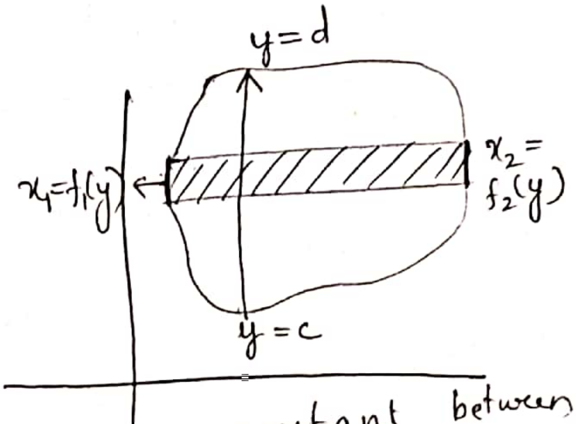
$$\iint_R f(x,y)\,dx\,dy = \int_a^b \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\; dx$$

First integrate $f(x,y)$ w.r.t 'y' treating 'x' as Constant between y_1 and y_2 then the resulting function is integrated w.r.t 'x' between the limits 'a' & 'b'.



Method II.

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\; dy$$



First integrated w.r.t x, Keeping y constant between the limits x_1 and x_2 and then the resulting expression is integrated w.r.t y between the limits 'c' and d.

III Method.

If the region R is a rectangle bounded by the lines $x=a$, $x=b$, $y=c$, $y=d$ then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_d^c f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$$

from which it is clear that the order of integration is immaterial for constant limits.

Problems

1. $\displaystyle\int_0^2 \int_1^2 (x^2+y^2)\,dx\,dy$, Order of integration is immaterial

$$\therefore I = \int_0^2 \left[\int_1^2 (x^2+y^2)\,dx \right] dy = \int_0^2 \left[\frac{x^3}{3} + y^2 x \right]_1^2 dy = \int_0^2 \left[\left(\frac{8}{3}+2y^2\right) - \left(\frac{1}{3}+y^2\right) \right] dy$$

$$= \int_0^2 \left(\frac{7}{3}+y^2\right) dy = \left[\frac{7}{3}y + \frac{y^3}{2} \right]_0^2 = \frac{14}{3} + \frac{8}{3} = \frac{22}{3}$$

2. $\displaystyle\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol^n

$$\int_{x=0}^1 \frac{1}{\sqrt{1-x^2}} \left\{ \int_{y=0}^1 \frac{dy}{\sqrt{1-y^2}} \right\} dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left\{ \sin^{-1} 1 - \sin^{-1} 0 \right\} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0 \right) dx = \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\pi}{2} \left\{ \sin^{-1} 1 - \sin^{-1} 0 \right\} = \frac{\pi^2}{4}$$

3. $\int_0^{\pi/4}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$.

Sol^n $\int_0^{\pi/4}\left[\int_0^{\pi/2} \sin(x+y)dx\right]dy = \int_0^{\pi/4}\left[-\cos(x+y)\right]_0^{\pi/2} dy$

$= -\int_0^{\pi/4}\left[\cos\left(\frac{\pi}{2}+y\right) - \cos(0+y)\right]dy$

$= -\int_0^{\pi/4}\left\{-\sin y - \cos y\right\}dy = \int_0^{\pi/4}\left\{\sin y + \cos y\right\}dy$

$= \left\{-\cos y + \sin y\right\}_0^{\pi/4} = \left[-\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right] - (-\cos 0 + \sin 0)$

$= \left(-1/\sqrt{2} + 1/\sqrt{2}\right) - (-1+0) = 1$

4. $\int_0^1\int_0^{\sqrt{1-y^2}} x^3 y\,dx\,dy = \int_0^1\left[\int_{x=0}^{x=\sqrt{1-y^2}} x^3 dx\right]y\,dy$

$= \int_0^1 y\left[\frac{x^4}{4}\right]_0^{\sqrt{1-y^2}} dy = \int_0^1 \frac{y}{4}\left\{(1-y^2)^2 - 0\right\}dy = \frac{1}{4}\int_0^1 y(1-y^2)^2\,dy$

$= \frac{1}{4}\int_0^1 y(1+y^4-2y^2)dy = \frac{1}{4}\int_0^1 (y+y^5-2y^3)\,dy$

$= 1/4\left[\frac{y^2}{2} + \frac{y^6}{6} - 2\frac{y^4}{4}\right]_0^1 = \frac{1}{4}\left(1/2 + 1/6 - 1/2\right) = \underline{\underline{1/24}}$

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5. Solve $\int_1^2 \int_3^4 (xy+e^y)\,dy\,dx$

Sol^n $\int_1^2 \left\{ \int_3^4 (xy+e^y)\,dy \right\} dx = \int_1^2 \left\{ \frac{xy^2}{2} + e^y \right\}_3^4 dx$

$= \int_1^2 \left\{ \left(\frac{x}{2}(16) + e^4 \right) - \left(\frac{x}{2}(9) + e^3 \right) \right\} dx = \int_1^2 \left(8x + e^4 - \frac{9x}{2} - e^3 \right) dx$

$= \int_1^2 \left\{ \frac{7}{2}x + (e^4 - e^3) \right\} dx = \left\{ \frac{7}{2} \cdot \frac{x^2}{2} + (e^4 - e^3) x \right\}_1^2$

$= \left\{ \frac{7}{4}x^2 + (e^4 - e^3)x \right\}_1^2 = \left(\frac{7}{4}(4) + (e^4 - e^3)2 \right) - \left(\frac{7}{4} + e^4 - e^3 \right)$

$= 7 + 2e^4 - 2e^3 - \frac{7}{4} - e^4 + e^3 = \frac{21}{4} + e^4 - e^3$

6. Solve $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$.

7. Solve $\int_1^4 \int_0^{\sqrt{4-x}} xy\,dy\,dx$

8. Solve $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

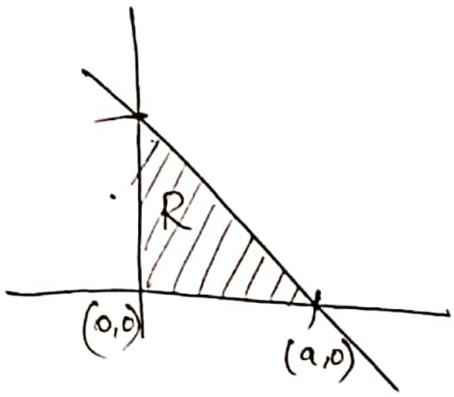
9. Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

10. Evaluate $\iint_R xy\, dx\, dy$ over the region in the positive quadrant for which $\frac{x}{a} + \frac{y}{b} \le 1$

Sol^n The given region is bounded by the x-axis, the y-axis and the line $\frac{x}{a} + \frac{y}{b} = 1$.

In this region 'x' varies from 0 to a and for each x, y varies from 0 to $b(1 - x/a)$

$$\iint_R xy\, dx\, dy = \int_0^a \int_{y=0}^{b(1-x/a)} xy\, dy\, dx$$



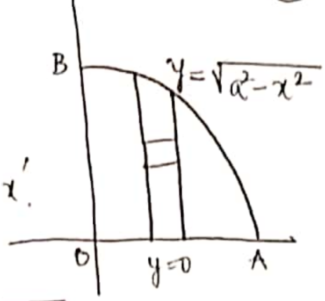
$$= \int_0^a x \left\{ \int_0^{b(1-x/a)} y\, dy \right\} dx$$

$$= \int_0^a x \left(\frac{y^2}{2}\right)_0^{b(1-x/a)} dx = \frac{b^2}{2} \int_0^a x \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{b^2}{2} \int_0^a \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2}\right) dx = \frac{b^2}{2}\left[\frac{x^2}{2} - \frac{2a^3}{3a} + \frac{a^4}{4a^2}\right] = \frac{a^2 b^2}{24}$$

11. Evaluate $\iint_R xy\, dx\, dy$ where R is the quadrant of the Circle $x^2 + y^2 = a^2$ where $x \geq 0$, $y \geq 0$.

Sol^n $\iint\limits_R xy\,dx\,dy$ $\left(x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}\right)$



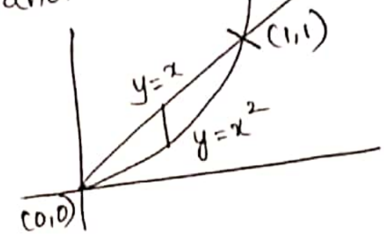
First integrate w.r.t 'y' & then w.r.t 'x'.

$$\iint\limits_R xy\,dx\,dy = \int_0^a\int_0^{\sqrt{a^2-x^2}} y\,dy\;x\,dx = \int_0^a x\,dx\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x\,dx\left[a^2-x^2-0\right] = \frac{1}{2}\int_0^a (xa^2-x^3)\,dx$$

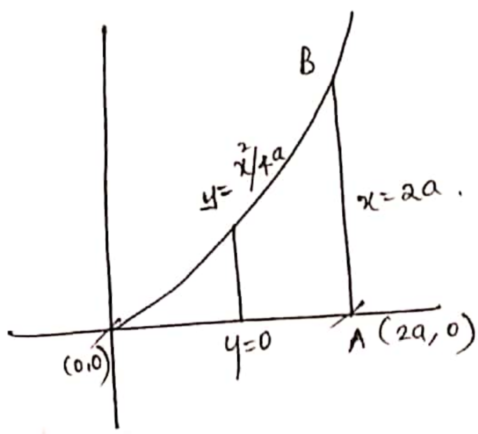
$$= \frac{1}{2}\left[\frac{x^2a^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{1}{2}\left[\left(\frac{a^4}{2}-\frac{a^4}{4}\right)\right] = \frac{a^4}{8}$$

12. Evaluate $\iint\limits_R xy(x+y)\,dx\,dy$ over the region 'R' bounded between the parabola $y=x^2$ and the line $y=x$



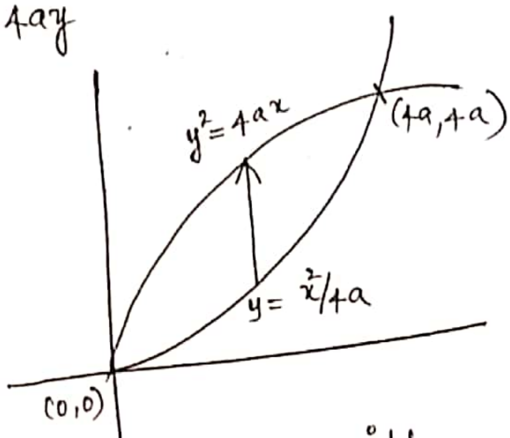
Sol^n $$\iint\limits_R xy(x+y)\,dx\,dy = \int_0^1\left[\int_{y=x^2}^{x} xy(x+y)\,dy\right]dx$$

13) Evaluate $\iint\limits_R xy\,dx\,dy$, 'R' is the region bounded by the x-axis, the ordinate $x=2a$ and the parabola $x^2=4ay$, $a>0$.



Sol^n $$\int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx.$$

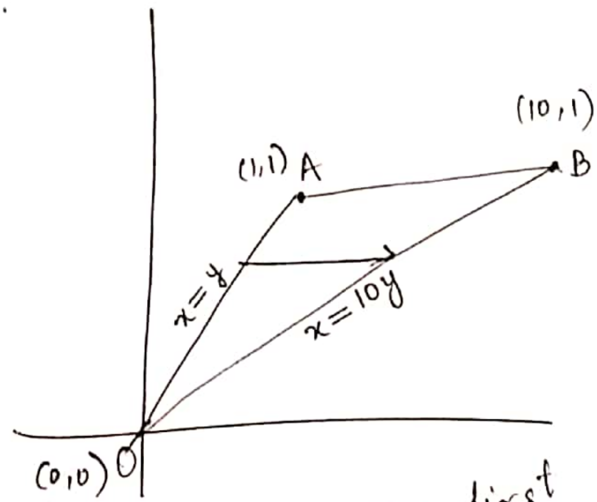
14) Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2=4ax$ and $x^2=4ay$



soln

$$\iint_R y\,dx\,dy = \int_0^{4a}\left[\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} y\,dy\right]dx$$

15) Evaluate $\iint_R \sqrt{xy-y^2}\,dy\,dx$ where 'R' is the triangle with vertices (0,0), (10,1) and (1,1).



soln

$0 \le y \le 1$

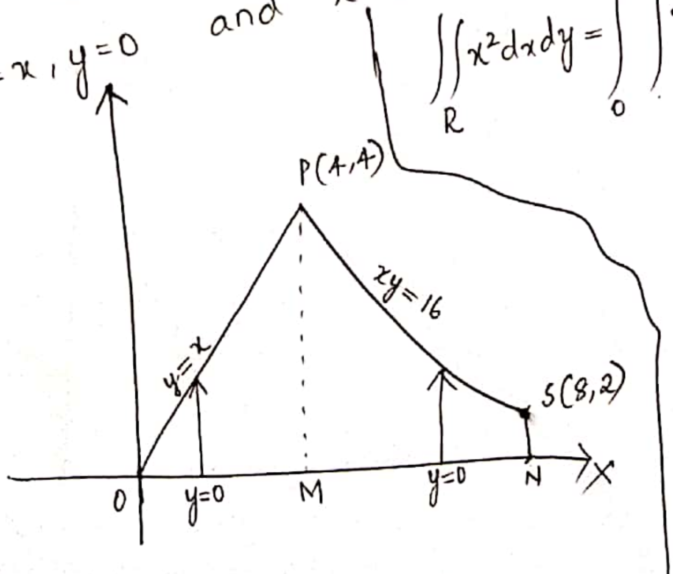
$y \le x \le 10y$

$$\iint_R \sqrt{xy-y^2}\,dx\,dy = \int_0^1\left[\int_{x=y}^{x=10y} \sqrt{xy-y^2}\,dx\right]dy$$

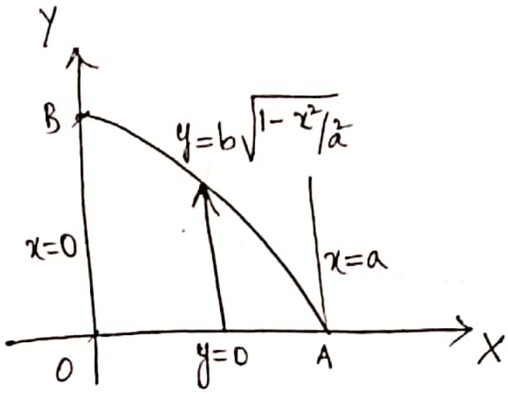
16) Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x$, $y=0$ and $x=8$.

$$\iint_R x^2\,dx\,dy = \int_0^4\int_0^x x^2\,dx\,dy + \int_4^8\int_0^{16/x} x^2\,dx\,dy$$

soln



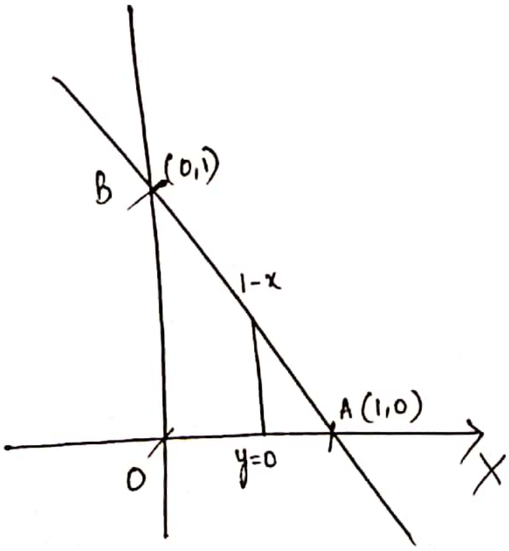
17) Evaluate $\iint_R xy\,dx\,dy$ over positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



sol^n $\iint_R xy\,dx\,dy = \iint_{OAB} xy\,dx\,dy = \int_{x=0}^{a}\left[\int_{y=0}^{b\sqrt{1-x^2/a^2}} y\,dy\right]dx$

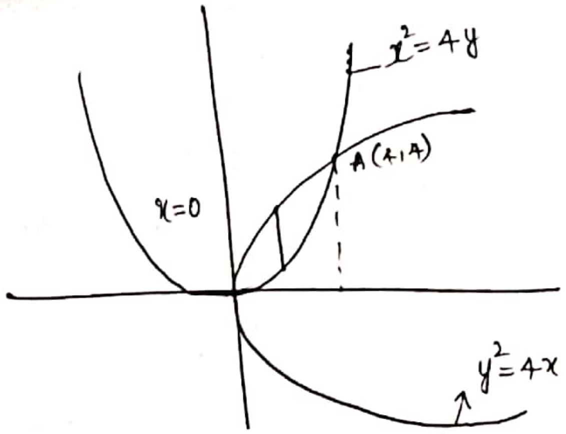
18) PT $\int_0^\infty \int_y^\infty x e^{-x^2/y}\,dx\,dy = \frac{1}{2}$

19) Evaluate $\iint_R e^{2x+3y}\,dx\,dy$ over the triangle bounded by $x=0$, $y=0$ and $x+y=1$



sol^n $\int_0^1 \int_0^{1-x} e^{2x+3y}\,dy\,dx$

20 Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$



soln Solving $y^2 = 4x$ and $x^2 = 4y$

We get, $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

Putting this value in $y^2 = 4x$ we get

$$\frac{x^4}{16} = 4x \Rightarrow x^4 = 64x$$

$$\Rightarrow x(x^3 - 64) = 0$$

$$\Rightarrow 0, 4$$

The pts are $(0,0)$ and $(4,4)$

$$\int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\,dx\,dy = \int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y\,dy \right] dx$$